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Generalized kinetic equations for a system of interacting atoms and photons: theory and simulations

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Abstract
In the present paper, we introduce generalized kinetic equations describing the dynamics of a system of interacting gas and photons obeying very general statistics. In the space homogeneous case, we study the equilibrium state of the system and investigate its stability by means of Lyapounov's theory. Two physically relevant situations are discussed in detail: photons in a background gas, and atoms in background radiation. After having dropped the statistics generalization for atoms but keeping the statistics generalization for photons, in the zero-order Chapmann–Enskog approximation, we present two numerical simulations where the system, initially at equilibrium, is perturbed by an external isotropic Dirac's delta and by a constant source of photons.

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1. Introduction

In several fields of nuclear and condensed matter physics, as well as in astrophysics and plasma physics, there exists experimental evidence which strongly suggests the necessity of introducing new statistics, different from the standard Boltzmann–Gibbs, Bose–Einstein and Fermi–Dirac ones.

Typically, generalized statistical distributions arise in systems exhibiting long-time microscopic memory, long-range interaction or fractal spacetime constraints [1–3]. Solar neutrinos [4], high energy nuclear collisions [5], and cosmic microwave background radiation [6, 7] are some typical phenomena where generalized statistical theories have been applied. Several models of q-deformed statistics have been used in the study of Bose gas condensation [8, 9], and phonon spectrum in ^{4}He [10], as well as in the study of asymmetric XXZ Heisenberg chain [11]. Moreover, quantum q-statistics has been introduced in the black-body theory [12, 13] where, based on an equilibrium statistical approach, a deformed Planck's law has been derived.

Recently, in [14], a generalized kinetic theory has been proposed by Rossani and Kaniadakis, with the purpose to deal with, at a kinetic level, particles obeying generalized statistics. Following this idea, generalized kinetic theories of electrons and photons [15] as well as electrons and phonons [16] have been proposed.

In this paper we present, as a natural continuation of the previous works, a kinetic theory of interacting atoms and photons obeying very general statistics.

We recall that in past years a classical kinetic approach in the study of this dynamical problem has been proposed for the case of two energy levels [17, 18] and further generalized for multi-level atoms and multi-frequency photons [19, 20]. The most remarkable feature is that this approach allows a self-consistent derivation, at equilibrium, of Planck's law.

The physical system we deal with is constituted by atoms A_ℓ ($\ell = 1, 2, \ldots, N$) with mass m, endowed with a finite number n of internal energy levels $0 = E_1 < E_2 < \cdots < E_n$, with transitions from one state to another made possible by either scattering between particles, or by their interaction with a self-consistent radiation field made up by photons p_{ij} with intensity I_{ij}, $i < j = 1, 2, \ldots, n$, at the $n(n-1)/2$ frequencies $\omega_{ij} = E_j - E_i$ (we adopt natural units $\hbar = c = 1$).

According to the relevant literature [21], we assume that the following interactions take place:

(a) elastic and inelastic interactions between particles,
(b) emission and absorption of photons.

We restrict ourselves to the most common and physical interaction mechanism between particles,

$$A_k + A_i \rightleftharpoons A_k + A_j \tag{1.1}$$

where $i \leqslant j$. The most general reaction [19] does not introduce significant improvement with respect to the present simplification.

Gas–radiation interaction processes include the following:

(i) Absorption:

$$A_i + p_{ij} \to A_j. \tag{1.2}$$

(ii) Spontaneous emission:

$$A_j \to A_i + p_{ij}. \tag{1.3}$$

(iii) Stimulated emission:

$$A_j + p_{ij} \to A_i + 2p_{ij}. \tag{1.4}$$

Following [17, 19], for these emission and absorption processes we assume the following: the outcoming atom has the same velocity as the incoming atom (that is, we neglect photon momentum with respect to atom momentum); photons are spontaneously emitted isotropically; photons which derive from stimulated emission have the same directions as the incoming photons.

Finally, gas–radiation interactions are modelled by means of the Einstein coefficients [22]: β_{ij} for both absorption and stimulated emission, α_{ij} for spontaneous emission.

The paper is organized as follows. In section 2, we introduce the kinetic equation for the physical system of interacting atoms and photons. In sections 3 and 4 theoretical results on equilibrium of the deformed model and its stability, by means of Lyapounov's theory, are given. The mathematical results are discussed on a physical ground, and connections with thermodynamics are pointed out. Two particular situations are studied in sections 5 and 6: photons in a background of atoms, and atoms in background radiation. In section 7, we drop the statistics generalization for atoms but keep the generalization for photons. The results of two numerical simulations are presented. In the first simulation, the system initially in equilibrium, is disturbed, starting from time $t = 0$, by a flash of monochromatic photons injected by an isotropic external source. In the second simulation, an isotropic source of monochromatic photons, constant in time, is inserted starting from time $t = 0$. Section 8 deals with the conclusions. Finally, in appendix A we present an overall summing up on the kinetic equations for a system of atoms and photons obeying standard statistics while, in appendix B we recall briefly two generalized statistical distributions used in the numerical simulations.

2. Generalized Boltzmann equations

In this section, we introduce the kinetic equation describing a system of atoms and photons obeying to a very general statistics. For the sake of clarity, we begin by introducing the Ühling–Uhlenbeck equation (UUe) in order to fix the notation used in the following. The UUe is a quantum extension of the Boltzmann equation [23]. Let us consider a system of particles with mass m which obeys the Bose–Einstein or Fermi–Dirac statistics. These particles interact elastically by means of binary encounters. In this situation, the UUe reads

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f = \int_{\mathbb{R}^3\times S^2} g\sigma(g,\zeta)\{f(\boldsymbol{v}')f(\boldsymbol{w}')[1+\lambda f(\boldsymbol{v})][1+\lambda f(\boldsymbol{w})] - f(\boldsymbol{v})f(\boldsymbol{w})[1+\lambda f(\boldsymbol{v}')][1+\lambda f(\boldsymbol{w}')]\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \tag{2.1}$$

where $\lambda = 1$ for bosons and $\lambda = -1$ for fermions. In equation (2.1) $f(\boldsymbol{v}) \equiv f(\boldsymbol{x},\boldsymbol{v},t)$ is the dimensionless distribution function, $\sigma(g,\zeta)$ is the cross section, with $g = |\boldsymbol{v}-\boldsymbol{w}|$ the relative speed. The post-collision velocities are given by

$$\boldsymbol{v}' = \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{w}+g\boldsymbol{\Omega}') \qquad \boldsymbol{w}' = \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{w}-g\boldsymbol{\Omega}') \tag{2.2}$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ are the velocities of the incoming particles and $\cos\zeta = \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'$, with $\boldsymbol{\Omega} = (\boldsymbol{v}-\boldsymbol{w})/g$ the unit vector in the direction of the relative speed. Finally, the two-dimensional unit sphere S^2 is the domain of integration for the unit vector $\boldsymbol{\Omega}'$.

In view of applications to particles which obey a more general statistics, we introduce the following substitution in the collision integral,

$$f(\boldsymbol{v}) \to \varphi[f(\boldsymbol{v})] \qquad 1+\lambda f(\boldsymbol{v}) \to \psi[f(\boldsymbol{v})] \tag{2.3}$$

where $\varphi(x)$ and $\psi(x)$ are non-negative functions supposed to be continuous as their derivatives. The limit values for $x \to 0$ are subject to precise physical requirement. For instance, $\varphi(0) = 0$ means that no transition occurs when one of the initial states is empty, $\psi(0) = 1$ means that no inhibition occurs when one of the final states is empty. According to [16], we assume that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varphi(x)}{\psi(x)}\right) > 0. \tag{2.4}$$

This is trivially true for bosons and fermions. In general, this assumption is justified *a posteriori* since it assures uniqueness and stability of equilibrium.

The generalized UUe now reads

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f = \int_{\mathbb{R}^3\times S^2} g\sigma(g,\zeta)\{\varphi[f(\boldsymbol{v}')]\varphi[f(\boldsymbol{w}')]\psi[f(\boldsymbol{v})]\psi[f(\boldsymbol{w})] - \varphi[f(\boldsymbol{v})]\varphi[f(\boldsymbol{w})]\psi[f(\boldsymbol{v}')]\psi[f(\boldsymbol{w}')]\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}'. \tag{2.5}$$

In the following, we specify equation (2.5) for a system of interacting atoms and photons. Let us introduce the characteristic departure and arrival functions [14]: $\varphi(f_\ell)$ and $\psi(f_\ell)$ for atoms, as well as $\Phi(I_{ij})$ and $\Psi(I_{ij})$ for photons, where atoms A_ℓ, endowed with energy level E_ℓ, are described by means of the distribution function $f_\ell(\boldsymbol{x}, \boldsymbol{v}, t)$, and photons p_{ij}, at frequency ω_{ij}, are described by means of the radiation intensity $I_{ij}(\boldsymbol{x}, \boldsymbol{\Omega}, t)$.

The distribution function of atoms A_ℓ obeys to the following system of generalized Boltzmann equations (see appendix A) [19],

$$\frac{\partial f_\ell}{\partial t} + \boldsymbol{v}\cdot\nabla f_\ell = J_\ell(\boldsymbol{v}) + \widetilde{J}_\ell(\boldsymbol{v}) \tag{2.6}$$

where on the right-hand side of equation (2.6) $J_\ell(\boldsymbol{v})$ describes the contribution due to the atom–atom interaction and $\widetilde{J}_\ell(\boldsymbol{v})$ the contribution due to the atom–photon interaction.

Explicitly, the collision integral, which takes into account the inelastic and elastic atom–atom collision, is given by

$$J_\ell(\boldsymbol{v}) = \sum_j \sum_{i\leqslant j} J^{(1)}_{ij\ell}(\boldsymbol{v}) + \sum_j \sum_{i\leqslant j} J^{(2)}_{ij\ell}(\boldsymbol{v}) + \sum_k \sum_{j\geqslant \ell} J^{(3)}_{jk\ell}(\boldsymbol{v}) + \sum_k \sum_{i\leqslant \ell} J^{(4)}_{ik\ell}(\boldsymbol{v}) \tag{2.7}$$

with

$$\begin{aligned} J^{(1)}_{ij\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} & g\sigma^{\ell i}_{\ell j}(g,\zeta)\big\{\varphi\big[f_\ell\big(\boldsymbol{v}^+_{ij}\big)\big]\varphi\big[f_i\big(\boldsymbol{w}^+_{ij}\big)\big]\psi[f_\ell(\boldsymbol{v})]\psi[f_j(\boldsymbol{w})] \\ & - \varphi[f_\ell(\boldsymbol{v})]\varphi[f_j(\boldsymbol{w})]\psi\big[f_\ell\big(\boldsymbol{v}^+_{ij}\big)\big]\psi\big[f_i\big(\boldsymbol{w}^+_{ij}\big)\big]\big\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \end{aligned} \tag{2.8}$$

$$\begin{aligned} J^{(2)}_{ij\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} & g\sigma^{\ell j}_{\ell i}(g,\zeta)\{\varphi[f_\ell(\boldsymbol{v}^-_{ij})]\varphi[f_j(\boldsymbol{w}^-_{ij})]\psi[f_\ell(\boldsymbol{v})]\psi[f_i(\boldsymbol{w})] \\ & - \varphi[f_\ell(\boldsymbol{v})]\varphi[f_i(\boldsymbol{w})]\psi[f_\ell(\boldsymbol{v}^-_{ij})]\psi[f_j(\boldsymbol{w}^-_{ij})]\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \end{aligned} \tag{2.9}$$

$$\begin{aligned} J^{(3)}_{jk\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} & g\sigma^{kj}_{k\ell}(g,\zeta)\{\varphi[f_j(\boldsymbol{v}^-_{\ell j})]\varphi[f_k(\boldsymbol{w}^-_{\ell j})]\psi[f_\ell(\boldsymbol{v})]\psi[f_k(\boldsymbol{w})] \\ & - \varphi[f_\ell(\boldsymbol{v})]\varphi[f_k(\boldsymbol{w})]\psi[f_j(\boldsymbol{v}^-_{\ell j})]\psi[f_k(\boldsymbol{w}^-_{\ell j})]\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \end{aligned} \tag{2.10}$$

$$\begin{aligned} J^{(4)}_{ik\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} & g\sigma^{ki}_{k\ell}(g,\zeta)\big\{\varphi\big[f_i\big(\boldsymbol{v}^+_{i\ell}\big)\big]\varphi\big[f_k\big(\boldsymbol{w}^+_{i\ell}\big)\big]\psi[f_\ell(\boldsymbol{v})]\psi[f_k(\boldsymbol{w})] \\ & - \varphi[f_\ell(\boldsymbol{v})]\varphi[f_k(\boldsymbol{w})]\psi\big[f_i\big(\boldsymbol{v}^+_{i\ell}\big)\big]\psi\big[f_k\big(\boldsymbol{w}^+_{i\ell}\big)\big]\big\}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \end{aligned} \tag{2.11}$$

where $\sigma^{\ell j}_{\ell i}$ and $\sigma^{\ell i}_{\ell j}$ are the cross sections for forward and backward reactions describing elastic and inelastic interactions. They satisfy the following microreversibility relationships [19]:

$$g^2\sigma^{\ell j}_{\ell i}(g,\boldsymbol{\Omega}') = (g^-_{ij})^2\sigma^{\ell i}_{\ell j}(g^-_{ij},\boldsymbol{\Omega}') \qquad g^2\sigma^{\ell i}_{\ell j}(g,\boldsymbol{\Omega}') = \big(g^+_{ij}\big)^2\sigma^{\ell j}_{\ell i}\big(g^+_{ij},\boldsymbol{\Omega}'\big). \tag{2.12}$$

The post-collision velocities are defined by

$$\boldsymbol{v}^\pm_{ij} = \tfrac{1}{2}\big(\boldsymbol{v} + \boldsymbol{w} + g^\pm_{ij}\boldsymbol{\Omega}'\big) \qquad \boldsymbol{w}^\pm_{ij} = \tfrac{1}{2}\big(\boldsymbol{v} + \boldsymbol{w} - g^\pm_{ij}\boldsymbol{\Omega}'\big) \tag{2.13}$$

with

$$g^\pm_{ij} = \sqrt{g^2 \pm \frac{4}{m}(E_j - E_i)}. \tag{2.14}$$

The four contributions to $J_\ell(\boldsymbol{v})$ correspond to the cases in which A_ℓ plays the role, in the reaction scheme (1.1), of A_k on the rhs, A_k on the lhs, A_i and A_j, respectively.

Otherwise, the collision integral $\widetilde{J}_\ell(\boldsymbol{v})$, which accounts for the gas–radiation interactions, is given by

$$\widetilde{J}_\ell(\boldsymbol{v}) = \sum_{i>\ell} \int_{S^2} \widehat{J}_{\ell i}(\boldsymbol{v}, \boldsymbol{\Omega})\, \mathrm{d}\boldsymbol{\Omega} - \sum_{i<\ell} \int_{S^2} \widehat{J}_{i\ell}(\boldsymbol{v}, \boldsymbol{\Omega})\, \mathrm{d}\boldsymbol{\Omega} \tag{2.15}$$

where

$$\widehat{J}_{i\ell}(\boldsymbol{v}, \boldsymbol{\Omega}) = \alpha_{i\ell}\{\varphi[f_\ell(\boldsymbol{v})]\Psi[I_{i\ell}(\boldsymbol{\Omega})]\psi[f_i(\boldsymbol{v})] - \varphi[f_i(\boldsymbol{v})]\Phi[I_{i\ell}(\boldsymbol{\Omega})]\psi[f_\ell(\boldsymbol{v})]\}. \tag{2.16}$$

By taking into account all the energy levels higher than ℓ, the loss term is due to absorption, while the gain term is due to spontaneous and stimulated emission. The situation is reversed when we consider all the energy levels lower than ℓ.

Finally, the kinetic equation for photons p_{ij} reads

$$\frac{\partial I_{ij}}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I_{ij} = \omega_{ij} \widetilde{J}_{ij}(\boldsymbol{\Omega}) \tag{2.17}$$

where

$$\widetilde{J}_{ij}(\boldsymbol{\Omega}) = \int_{\mathbb{R}^3} \widehat{J}_{ij}(\boldsymbol{v}, \boldsymbol{\Omega})\, \mathrm{d}\boldsymbol{v}. \tag{2.18}$$

Again, the gain term is due to spontaneous and stimulated emission, while the loss term is due to absorption.

It is easy to see that, by posing

$$\varphi[f_\ell(\boldsymbol{v})] \to f_\ell(\boldsymbol{v}) \qquad \psi[f_\ell(\boldsymbol{v})] \to 1 \tag{2.19}$$

$$\Phi[I_{ij}(\boldsymbol{\Omega})] \to \frac{\beta_{ij}}{\alpha_{ij}} I_{ij}(\boldsymbol{\Omega}) \qquad \Psi[I_{ij}(\boldsymbol{\Omega})] \to 1 + \frac{\beta_{ij}}{\alpha_{ij}} I_{ij}(\boldsymbol{\Omega}) \tag{2.20}$$

equations (2.6) and (2.17) reduce to the standard kinetic equations for atoms and photons [19, 20].

From now on, for the sake of simplicity, we adopt the notation $\varphi_\ell(\boldsymbol{v}) \equiv \varphi[f_\ell(\boldsymbol{v})]$, $\psi_\ell(\boldsymbol{v}) \equiv \psi[f_\ell(\boldsymbol{v})]$, $\Phi_{ij}(\boldsymbol{\Omega}) \equiv \Phi[I_{ij}(\boldsymbol{\Omega})]$ and $\Psi_{ij}(\boldsymbol{\Omega}) \equiv \Psi[I_{ij}(\boldsymbol{\Omega})]$.

3. Equilibrium

Notwithstanding, the generalized equations are more complicate with respect to the classical kinetic equations, many of the methods used in the standard kinetic theory are still applicable. This is, in particular, true for the study of equilibria and their stability.

Lemma 1. *For any given arbitrary smooth functions $\gamma_\ell(\boldsymbol{v})$ and $\Gamma_{ij}(\boldsymbol{\Omega})$ the following two relations hold:*

$$\begin{aligned}\sum_\ell \int_{\mathbb{R}^3} \gamma_\ell(\boldsymbol{v}) J_\ell(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = \sum_\ell \sum_j \sum_{i\leqslant j} &\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} g\sigma_{\ell j}^{\ell i}(g, \zeta) \\ &\times \left[\varphi_\ell\left(\boldsymbol{v}_{ij}^+\right)\varphi_i\left(\boldsymbol{w}_{ij}^+\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) - \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}_{ij}^+\right)\psi_i\left(\boldsymbol{w}_{ij}^+\right)\right] \\ &\times \left[\gamma_\ell(\boldsymbol{v}) + \gamma_j(\boldsymbol{w}) - \gamma_\ell\left(\boldsymbol{v}_{ij}^+\right) - \gamma_i\left(\boldsymbol{w}_{ij}^+\right)\right] \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{w}\, \mathrm{d}\boldsymbol{\Omega}'\end{aligned} \tag{3.1}$$

$$\begin{aligned}\sum_\ell \int_{\mathbb{R}^3} \gamma_\ell(\boldsymbol{v}) \widetilde{J}_\ell(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} + \sum_j \sum_{i<j} \int_{S^2} \Gamma_{ij}(\boldsymbol{\Omega}) \widetilde{J}_{ij}(\boldsymbol{\Omega})\, \mathrm{d}\boldsymbol{\Omega} \\ = \sum_j \sum_{i<j} \alpha_{ij} \int_{\mathbb{R}^3\times S^2} [\varphi_j(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) - \varphi_i(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})] \\ \times [\gamma_i(\boldsymbol{v}) + \Gamma_{ij}(\boldsymbol{\Omega}) - \gamma_j(\boldsymbol{v})]\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{\Omega}.\end{aligned} \tag{3.2}$$

Proof. The proof for the generalized case follows the same steps which can be found in [19] for the standard theory. In the following, we present the main points recalling the relevant papers for the details [19, 24, 25].

Equation (3.1) follows from the definition of the collision integral $J_\ell(\boldsymbol{v})$ given in equation (2.7). The relevant addends may be grouped together, with suitable interchange of the indices i, j, k and ℓ, to form the collision term appearing on the rhs of equation (3.1). In particular, for all the $J^{(2)}_{ij\ell}(\boldsymbol{v})$ and $J^{(3)}_{jk\ell}(\boldsymbol{v})$ the microreversibility relationships (2.12) are used as well as the relation

$$\frac{g^{\pm}_{ij}}{g}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' = \mathrm{d}\boldsymbol{v}^{\pm}_{ij}\,\mathrm{d}\boldsymbol{w}^{\pm}_{ij}\,\mathrm{d}\boldsymbol{\Omega} \tag{3.3}$$

where the Jacobian of the transformation in equation (3.3) arises from the definitions (2.13). In the same way, equation (3.2) is obtained from the collision integrals $\widetilde{J}_\ell(\boldsymbol{v})$ and $\widetilde{J}_{ij}(\boldsymbol{\Omega})$ given in equations (2.15) and (2.18), after suitable interchange of the indices i, j and ℓ. □

In the space homogeneous case, where all the dynamical quantities f_ℓ and I_{ij} are functions only of the time, equilibrium is defined by

$$\frac{\partial f_\ell}{\partial t} = \frac{\partial I_{ij}}{\partial t} = 0. \tag{3.4}$$

Let us introduce the following functional:

$$\mathcal{D} = \sum_\ell \int_{\mathbb{R}^3} \ln\left(\frac{\varphi_\ell}{\psi_\ell}\right)\frac{\partial f_\ell}{\partial t}\,\mathrm{d}\boldsymbol{v} + \sum_j\sum_{i<j}\frac{1}{\omega_{ij}}\int_{S^2}\ln\left(\frac{\Phi_{ij}}{\Psi_{ij}}\right)\frac{\partial I_{ij}}{\partial t}\,\mathrm{d}\boldsymbol{\Omega}. \tag{3.5}$$

After using the kinetic equations (2.6) and (2.17), equation (3.5) becomes

$$\mathcal{D} = \sum_\ell \int_{\mathbb{R}^3} \ln\left(\frac{\varphi_\ell}{\psi_\ell}\right) J_\ell(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} + \sum_\ell \int_{\mathbb{R}^3} \ln\left(\frac{\varphi_\ell}{\psi_\ell}\right)\widetilde{J}_\ell(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} + \sum_j\sum_{i<j}\int_{S^2}\ln\left(\frac{\Phi_{ij}}{\Psi_{ij}}\right)\widetilde{J}_{ij}(\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} \tag{3.6}$$

and by applying lemma 1, with $\gamma_\ell = \ln(\varphi_\ell/\psi_\ell)$ and $\Gamma_{ij} = \ln(\Phi_{ij}/\Psi_{ij})$, we obtain

$$\begin{aligned}\mathcal{D} = {} & \sum_\ell\sum_j\sum_{i\leqslant j}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} g\sigma^{\ell i}_{\ell j}(g,\zeta) \\ & \times\left[\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) - \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right)\right] \\ & \times\ln\frac{\varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right)}{\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w})}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \\ & + \sum_j\sum_{i<j}\alpha_{ij}\int_{\mathbb{R}^3\times S^2}[\varphi_j(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) - \varphi_i(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})] \\ & \times\ln\frac{\varphi_i(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})}{\varphi_j(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v})}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{\Omega}\end{aligned} \tag{3.7}$$

which is a quantity manifestly not positive since $(A-B)\ln(B/A) \leqslant 0$.

Proposition 1. *Condition (3.4) is equivalent to the following couple of equations:*

$$\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) = \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right) \tag{3.8}$$

$$\varphi_j(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) = \varphi_i(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v}). \tag{3.9}$$

Proof. Taking into account the definitions of the collision integral given in equations (2.7), (2.15) and (2.18), we observe first that equations (3.8) and (3.9) imply $J_\ell(\boldsymbol{v}) = \widetilde{J}_\ell(\boldsymbol{v}) = \widetilde{J}_{ij}(\boldsymbol{\Omega}) = 0$ and thus, from the kinetic equations (2.6) and (2.17), we obtain condition (3.4).

On the other hand, from equation (3.5) by using equation (3.4), it follows that $\mathcal{D} = 0$. But, since in equation (3.7) both the integrands are never positive, condition $\mathcal{D} = 0$ implies equations (3.8) and (3.9). □

In the case $i = j$, equation (3.8) gives

$$\ln\left(\frac{\varphi_\ell(\boldsymbol{v})}{\psi_\ell(\boldsymbol{v})}\right) + \ln\left(\frac{\varphi_i(\boldsymbol{w})}{\psi_i(\boldsymbol{w})}\right) = \ln\left(\frac{\varphi_\ell\left(\boldsymbol{v}_{ii}^+\right)}{\psi_\ell\left(\boldsymbol{v}_{ii}^+\right)}\right) + \ln\left(\frac{\varphi_i\left(\boldsymbol{w}_{ii}^+\right)}{\psi_i\left(\boldsymbol{w}_{ii}^+\right)}\right) \tag{3.10}$$

therefore $\ln(\varphi_\ell/\psi_\ell)$ is a collision invariant for atoms, that is

$$\ln\left(\frac{\varphi_i}{\psi_i}\right) = -\frac{1}{T}\left(\frac{1}{2}m\boldsymbol{v}^2 + E_i - \mu_i\right) \tag{3.11}$$

where T is the absolute temperature of the whole system of atoms and photons. (Here and in the following we pose $k_{\mathrm{B}} = 1$.)

For the case $i \neq j$, from equation (3.8) and by using equation (3.11), we get, as additional condition, that all the chemical potentials are equal: $\mu_i = \mu_j \equiv \mu$.

Otherwise, from equation (3.9), using equation (3.11), follows

$$\ln\left(\frac{\Phi_{ij}}{\Psi_{ij}}\right) = -\frac{\omega_{ij}}{T} \tag{3.12}$$

which is the generalized version of Planck's law. In appendix B we write explicitly equation (3.12) for two particular statistics.

Let us observe that, due to the monotonicity of both φ_ℓ/ψ_ℓ and Φ_{ij}/Ψ_{ij}, equations (3.11) and (3.12) give a unique equilibrium solution for $f_\ell(\boldsymbol{v})$ and $I_{ij}(\boldsymbol{\Omega})$, respectively.

4. Stability

In order to study the stability of such equilibrium solution let us introduce the following functional,

$$L = H_{\mathrm{A}} + H_{\mathrm{p}} \tag{4.1}$$

where

$$H_{\mathrm{A}} = \sum_\ell \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} \tag{4.2}$$

$$H_{\mathrm{p}} = \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij})\,\mathrm{d}\boldsymbol{\Omega} \tag{4.3}$$

with

$$\frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell} = \ln\left(\frac{\varphi_\ell}{\psi_\ell}\right) \tag{4.4}$$

$$\frac{\partial \mathcal{H}_{\mathrm{p}}(I_{ij})}{\partial I_{ij}} = \ln\left(\frac{\Phi_{ij}}{\Psi_{ij}}\right). \tag{4.5}$$

Remark that, since φ_ℓ/ψ_ℓ and Φ_{ij}/Ψ_{ij} have been assumed to be monotonically increasing, $\mathcal{H}_{\mathrm{A}}(f_\ell)$ and $\mathcal{H}_{\mathrm{p}}(I_{ij})$ are convex functions of their arguments.

From [14] we know that $S = -L$ is nothing else but entropy density for the present physical system and it is the sum of two contributions: gas entropy and photon entropy [21, 24, 26].

Lemma 2. *The condition*

$$\sum_\ell \int_{\mathbb{R}^3} \left(\frac{\partial \mathcal{H}_\mathrm{A}(f_\ell)}{\partial f_\ell}\right)^* (f_\ell - f_\ell^*)\,\mathrm{d}\boldsymbol{v} + \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \left(\frac{\partial \mathcal{H}_\mathrm{p}(I_{ij})}{\partial I_{ij}}\right)^* (I_{ij} - I_{ij}^*)\,\mathrm{d}\boldsymbol{\Omega} = 0 \tag{4.6}$$

where $*$ means at equilibrium, is equivalent to energy conservation $E(f_\ell, I_{ij}) = E^*(f_\ell^*, I_{ij}^*)$.

Proof. Energy can be written as

$$E = E_\mathrm{A} + E_\mathrm{p} \tag{4.7}$$

where

$$E_\mathrm{A} = \int_{\mathbb{R}^3} \mathcal{E}_\mathrm{A}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \tag{4.8}$$

$$E_\mathrm{p} = \int_{S^2} \mathcal{E}_\mathrm{p}(\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} \tag{4.9}$$

with

$$\mathcal{E}_\mathrm{A} = \sum_\ell \left(\frac{1}{2} m \boldsymbol{v}^2 + E_\ell\right) f_\ell(\boldsymbol{v}) \tag{4.10}$$

$$\mathcal{E}_\mathrm{p} = \sum_j \sum_{i<j} I_{ij}(\boldsymbol{\Omega}). \tag{4.11}$$

By means of Euler's theorem we can write

$$E - E^* = \sum_\ell \int_{\mathbb{R}^3} \left(\frac{\partial \mathcal{E}_\mathrm{A}}{\partial f_\ell}\right)^* (f_\ell - f_\ell^*)\,\mathrm{d}\boldsymbol{v} + \sum_j \sum_{i<j} \int_{S^2} \left(\frac{\partial \mathcal{E}_\mathrm{p}}{\partial I_{ij}}\right)^* (I_{ij} - I_{ij}^*)\,\mathrm{d}\boldsymbol{\Omega}. \tag{4.12}$$

Otherwise, by using equations (3.11) and (3.12), from equations (4.4) and (4.5), the following relationships are easily obtained:

$$\left(\frac{\partial \mathcal{H}_\mathrm{A}(f_\ell)}{\partial f_\ell}\right)^* = \frac{1}{T^*}\left[\mu^* - \left(\frac{\partial \mathcal{E}_\mathrm{A}}{\partial f_\ell}\right)^*\right] \tag{4.13}$$

$$\left(\frac{\partial \mathcal{H}_p\left(I_{ij}\right)}{\partial I_{ij}}\right)^* = -\frac{\omega_{ij}}{T^*}\left(\frac{\partial \mathcal{E}_\mathrm{p}}{\partial I_{ij}}\right)^*. \tag{4.14}$$

From equation (4.12), by taking into account equations (4.13), (4.14) and particle conservation $\sum_\ell \int f_\ell\,\mathrm{d}\boldsymbol{v} = \sum_\ell \int f_\ell^*\,\mathrm{d}\boldsymbol{v}$, one easily realizes that $E = E^*$ implies equation (4.6). □

The main result with respect to stability can be summarized as follows:

Proposition 2. *The functional L is a strict Lyapounov functional for the present dynamical system: the unique equilibrium is asymptotically stable, and any initial state will relax to it asymptotically in time.*

Proof. First, one easily proves that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \mathcal{D} \leqslant 0 \tag{4.15}$$

with $\mathrm{d}L/\mathrm{d}t = 0$ only at the unique equilibrium position. In fact, from the definition (4.1) it follows that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_{\ell} \int_{\mathbb{R}^3} \frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell} \frac{\partial f_\ell}{\partial t} \,\mathrm{d}\boldsymbol{v} + \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \frac{\partial \mathcal{H}_{\mathrm{p}}(I_{ij})}{\partial I_{ij}} \frac{\partial I_{ij}}{\partial t} \,\mathrm{d}\boldsymbol{\Omega} \tag{4.16}$$

and taking into account equations (4.4) and (4.5), from the definition (3.5) follows equation (4.15). On the other hand, by taking into account lemma 2, one has

$$\begin{aligned}
L - L^* &= H_{\mathrm{A}} - H_{\mathrm{A}}^* + H_{\mathrm{p}} - H_{\mathrm{p}}^* \\
&= \sum_{\ell} \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} - \sum_{\ell} \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell^*)\,\mathrm{d}\boldsymbol{v} \\
&\quad + \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij})\,\mathrm{d}\boldsymbol{\Omega} - \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij}^*)\,\mathrm{d}\boldsymbol{\Omega} \\
&= \sum_{\ell} \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} - \sum_{\ell} \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell^*)\,\mathrm{d}\boldsymbol{v} - \sum_{\ell} \int_{\mathbb{R}^3} \left(\frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell} \right)^* (f_\ell - f_\ell^*)\,\mathrm{d}\boldsymbol{v} \\
&\quad + \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij})\,\mathrm{d}\boldsymbol{\Omega} - \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij}^*)\,\mathrm{d}\boldsymbol{\Omega} \\
&\quad - \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \left(\frac{\partial \mathcal{H}_{\mathrm{p}}(I_{ij})}{\partial I_{ij}} \right)^* (I_{ij} - I_{ij}^*)\,\mathrm{d}\boldsymbol{\Omega} \\
&= \sum_{\ell} \int_{\mathbb{R}^3} \widehat{\mathcal{H}}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} + \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \widehat{\mathcal{H}}_{\mathrm{p}}(I_{ij})\,\mathrm{d}\boldsymbol{\Omega}
\end{aligned} \tag{4.17}$$

where

$$\widehat{\mathcal{H}}_{\mathrm{A}}(f_\ell) = \mathcal{H}_{\mathrm{A}}(f_\ell) - \left[\mathcal{H}_{\mathrm{A}}(f_\ell^*) + \left(\frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell} \right)^* (f_\ell - f_\ell^*) \right] \tag{4.18}$$

$$\widehat{\mathcal{H}}_{\mathrm{p}}(I_{ij}) = \mathcal{H}_{\mathrm{p}}(I_{ij}) - \left[\mathcal{H}_{\mathrm{p}}(I_{ij}^*) + \left(\frac{\partial \mathcal{H}_{\mathrm{p}}(I_{ij})}{\partial I_{ij}} \right)^* (I_{ij} - I_{ij}^*) \right]. \tag{4.19}$$

Since both $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{p}}$ are convex, we have $\widehat{\mathcal{H}}_{\mathrm{A}} \geqslant 0$ and $\widehat{\mathcal{H}}_{\mathrm{p}} \geqslant 0$, and then we can conclude that $L \geqslant L^*$ and attains its minimum, $L = L^*$, only for the unique equilibrium position. □

In the next two sections, we inquire on equilibrium and stability in two particular cases: photons in a background gas and atoms in a background radiation.

5. Photons in a background gas

An usual assumption in radiation gasdynamics is that the relaxation time due to atom–atom interactions is much quicker than that due to gas–radiation processes. Thus, in the kinetic equation for photons we can fix the distribution of atoms as an equilibrium function f_ℓ^* at a certain temperature T.

In order to study equilibrium and its stability, we introduce the following functional

$$\mathcal{C}_{\rm p} = \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \ln\left[\frac{\Phi_{ij}}{\Psi_{ij}} \exp\left(\frac{\omega_{ij}}{T}\right)\right]\left(\frac{\partial I_{ij}}{\partial t}\right)^{\diamond} {\rm d}\boldsymbol{\Omega} \tag{5.1}$$

where $\diamond$ means that f_ℓ has been substituted by f_ℓ^*.

After using the kinetic equation (2.17), equation (5.1) becomes

$$\mathcal{C}_{\rm p} = \sum_j \sum_{i<j} \int_{S^2} \ln\left[\frac{\Phi_{ij}}{\Psi_{ij}} \exp\left(\frac{\omega_{ij}}{T}\right)\right][\widetilde{J}_{ij}(\boldsymbol{\Omega})]^{\diamond} {\rm d}\boldsymbol{\Omega} \tag{5.2}$$

and using lemma 1 with all the $\gamma_\ell(\boldsymbol{v}) = 0$ and $\Gamma_{ij}(\boldsymbol{\Omega}) = \ln[(\Phi_{ij}/\Psi_{ij})\exp(\omega_{ij}/T)]$ we obtain

$$\mathcal{C}_{\rm p} = \sum_j \sum_{i<j} \alpha_{ij} \int_{S^2} [\varphi_j^*(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i^*(\boldsymbol{v}) - \varphi_i^*(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j^*(\boldsymbol{v})] \ln\left[\frac{\Phi_{ij}}{\Psi_{ij}} \exp\left(\frac{\omega_{ij}}{T}\right)\right] {\rm d}\boldsymbol{v}\, {\rm d}\boldsymbol{\Omega}. \tag{5.3}$$

Finally, observing that

$$\omega_{ij} = \left(\tfrac{1}{2}m\boldsymbol{v}^2 + E_j - \mu\right) - \left(\tfrac{1}{2}m\boldsymbol{v}^2 + E_i - \mu\right) \tag{5.4}$$

and taking into account equation (3.11), the quantity $\mathcal{C}_{\rm p}$ becomes

$$\mathcal{C}_{\rm p} = \sum_j \sum_{i<j} \alpha_{ij} \int_{S^2} [\varphi_j^*(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i^*(\boldsymbol{v}) - \varphi_i^*(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j^*(\boldsymbol{v})] \ln \frac{\varphi_i^*(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j^*(\boldsymbol{v})}{\varphi_j^*(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i^*(\boldsymbol{v})} {\rm d}\boldsymbol{\Omega} \tag{5.5}$$

which is manifestly not positive.

From usual arguments (see proof of proposition 1), the equilibrium condition $\partial I_{ij}/\partial t = 0$ is equivalent to

$$\varphi_j^*(\boldsymbol{v})\Psi_{ij}(\boldsymbol{\Omega})\psi_i^*(\boldsymbol{v}) = \varphi_i^*(\boldsymbol{v})\Phi_{ij}(\boldsymbol{\Omega})\psi_j^*(\boldsymbol{v}). \tag{5.6}$$

Again, after using equation (3.11), from equation (5.6) we find the generalized Planck's law (3.12) for photons.

In order to investigate the stability of such equilibrium, let us introduce the following functional:

$$L_{\rm p} = H_{\rm p} + \frac{E_{\rm p}}{T} \tag{5.7}$$

where $H_{\rm p}$ is defined in equation (4.3) and $E_{\rm p}$ is given in equation (4.9).

Proposition 3. *$L_{\rm p}$ is a Lyapounov functional for the present problem.*

Proof. First of all we find

$$\frac{{\rm d}L_{\rm p}}{{\rm d}t} = \mathcal{C}_{\rm p} \leqslant 0 \tag{5.8}$$

thus $L_{\rm p}$ is a decreasing function. In fact, from the definition (5.7) we have

$$\frac{{\rm d}L_{\rm p}}{{\rm d}t} = \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \left[\frac{\partial \mathcal{H}_{\rm p}(I_{ij})}{\partial I_{ij}} + \frac{\omega_{ij}}{T}\right]\left(\frac{\partial I_{ij}}{\partial t}\right)^{\diamond} {\rm d}\boldsymbol{\Omega} \tag{5.9}$$

and after using equation (4.5) we obtain equation (5.8).

Moreover, since

$$\left(\frac{\partial \mathcal{H}_{\rm p}(I_{ij})}{\partial I_{ij}}\right)^* = -\frac{\omega_{ij}}{T} \tag{5.10}$$

as it follows from equations (3.12) and (4.5), we can write

$$
\begin{aligned}
L_{\mathrm{p}} - L_{\mathrm{p}}^* &= H_{\mathrm{p}} - H_{\mathrm{p}}^* + \frac{1}{T}(E_{\mathrm{p}} - E_{\mathrm{p}}^*) \\
&= \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij})\, \mathrm{d}\boldsymbol{\Omega} - \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \mathcal{H}_{\mathrm{p}}(I_{ij}^*)\, \mathrm{d}\boldsymbol{\Omega} \\
&\quad + \frac{1}{T} \sum_j \sum_{i<j} \int_{S^2} (I_{ij} - I_{ij}^*)\, \mathrm{d}\boldsymbol{\Omega} \\
&= \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \left[\mathcal{H}_{\mathrm{p}}(I_{ij}) - \mathcal{H}_{\mathrm{p}}(I_{ij}^*) - \left(\frac{\partial \mathcal{H}_{\mathrm{p}}(I_{ij})}{\partial I_{ij}} \right)^* (I_{ij} - I_{ij}^*) \right] \mathrm{d}\boldsymbol{\Omega} \\
&= \sum_j \sum_{i<j} \frac{1}{\omega_{ij}} \int_{S^2} \widehat{\mathcal{H}}_{\mathrm{p}}(I_{ij})\, \mathrm{d}\boldsymbol{\Omega}
\end{aligned} \tag{5.11}
$$

where $\widehat{\mathcal{H}}_{\mathrm{p}}$ is given in equation (4.19). Because $\mathcal{H}_{\mathrm{p}}$ is convex we conclude that $L_{\mathrm{p}} \geqslant L_{\mathrm{p}}^*$, that is L_{p} attains a minimum at equilibrium. □

Taking into account of the definition of L_{p} it is easy to realize that equation (5.8) is equivalent to Clausius inequality

$$
\frac{\mathrm{d}S_{\mathrm{p}}}{\mathrm{d}t} \geqslant \frac{1}{T} \frac{\mathrm{d}E_{\mathrm{p}}}{\mathrm{d}t} \tag{5.12}
$$

where $S_{\mathrm{p}} = -H_{\mathrm{p}}$. From $L_{\mathrm{p}} - L_{\mathrm{p}}^* \geqslant 0$ it follows that the quantity $\mathcal{S}_{\mathrm{p}} = S_{\mathrm{p}} - E_{\mathrm{p}}/T$ is always increasing and attains a maximum at equilibrium.

6. Atoms in a background radiation

Suppose now that temperature is high enough so that we can consider photons as an equilibrium background. This means that, in the kinetic equations for atoms, we can fix the distribution of photons as an equilibrium function I_{ij}^* at a certain temperature $\mathcal{T}$.

In order to study equilibrium and its stability, we introduce the following functional,

$$
\mathcal{C}_{\mathrm{A}} = \sum_\ell \int_{\mathbb{R}^3} \ln\left\{ \frac{\varphi_\ell}{\psi_\ell} \exp\left[\frac{1}{T} \left(\frac{1}{2} m \boldsymbol{v}^2 + E_\ell - \mu \right) \right] \right\} \left(\frac{\partial f_\ell}{\partial t} \right)^\sharp \mathrm{d}\boldsymbol{v} \tag{6.1}
$$

where $\sharp$ means that I_{ij} has been substituted by I_{ij}^*. After using the kinetic equation (2.6), equation (6.1) becomes

$$
\mathcal{C}_{\mathrm{A}} = \sum_\ell \int_{\mathbb{R}^3} \ln\left\{ \frac{\varphi_\ell}{\psi_\ell} \exp\left[\frac{1}{T} \left(\frac{1}{2} m \boldsymbol{v}^2 + E_\ell - \mu \right) \right] \right\} [J_\ell(\boldsymbol{v}) + \widetilde{J}_\ell(\boldsymbol{v})]^\sharp\, \mathrm{d}\boldsymbol{v}. \tag{6.2}
$$

Using lemma 1 with $\gamma_\ell(\boldsymbol{v}) = \ln\{(\varphi_\ell/\psi_\ell)\exp[(m\boldsymbol{v}^2/2 + E_\ell - \mu)/T]\}$ and all $\Gamma_{ij}(\boldsymbol{\Omega}) = 0$, equation (6.2) becomes

$$
\begin{aligned}
\mathcal{C}_{\mathrm{A}} = \sum_\ell \sum_j \sum_{i\leqslant j} \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} & g\sigma_{\ell j}^{\ell i}(g,\zeta) \big[\varphi_\ell\big(\boldsymbol{v}_{ij}^+\big)\varphi_i\big(\boldsymbol{w}_{ij}^+\big)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) \\
& - \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\big(\boldsymbol{v}_{ij}^+\big)\psi_i\big(\boldsymbol{w}_{ij}^+\big)\big] \ln\left\{ \frac{\varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\big(\boldsymbol{v}_{ij}^+\big)\psi_i\big(\boldsymbol{w}_{ij}^+\big)}{\varphi_\ell\big(\boldsymbol{v}_{ij}^+\big)\varphi_i\big(\boldsymbol{w}_{ij}^+\big)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w})} \right. \\
& \left. \times \exp\left[\frac{E_j - E_i}{T} - \frac{m}{2T}\Big(\big(\boldsymbol{v}_{ij}^+\big)^2 + \big(\boldsymbol{w}_{ij}^+\big)^2 - \boldsymbol{v}^2 - \boldsymbol{w}^2\Big) \right] \right\} \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{w}\, \mathrm{d}\boldsymbol{\Omega}'
\end{aligned}
$$

$$+\sum_j \sum_{i<j} \alpha_{ij} \int_{\mathbb{R}^3\times S^2} [\varphi_j(\boldsymbol{v})\Psi^*_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) - \varphi_i(\boldsymbol{v})\Phi^*_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})]$$
$$\times \ln\left[\frac{\varphi_i(\boldsymbol{v})\psi_j(\boldsymbol{v})}{\varphi_j(\boldsymbol{v})\psi_i(\boldsymbol{v})} \exp\left(-\frac{E_j - E_i}{T}\right)\right] \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{\Omega} \tag{6.3}$$

and taking into account the energy conservation and equation (3.12)

$$\omega_{ij} = E_j - E_i = \tfrac{1}{2}m\left[\left(\boldsymbol{v}^+_{ij}\right)^2 + \left(\boldsymbol{w}^+_{ij}\right)^2 - \boldsymbol{v}^2 - \boldsymbol{w}^2\right] \tag{6.4}$$

we obtain

$$\mathcal{C}_{\mathrm{A}} = \sum_\ell \sum_j \sum_{i\leqslant j} \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} g\sigma^{\ell i}_{\ell j}(g,\zeta)$$
$$\times\left[\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) - \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right)\right]$$
$$\times \ln \frac{\varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right)}{\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w})} \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}'$$
$$+\sum_j \sum_{i<j} \alpha_{ij} \int_{\mathbb{R}^3\times S^2} \left[\varphi_j(\boldsymbol{v})\Psi^*_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) - \varphi_i(\boldsymbol{v})\Phi^*_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})\right]$$
$$\times \ln \frac{\varphi_i(\boldsymbol{v})\Phi^*_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v})}{\varphi_j(\boldsymbol{v})\Psi^*_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v})} \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{\Omega} \tag{6.5}$$

which is manifestly non-positive.

Following the same steps given in proposition 1, we obtain that the equilibrium condition $\partial f_\ell/\partial t = 0$ is equivalent to

$$\varphi_\ell\left(\boldsymbol{v}^+_{ij}\right)\varphi_i\left(\boldsymbol{w}^+_{ij}\right)\psi_\ell(\boldsymbol{v})\psi_j(\boldsymbol{w}) = \varphi_\ell(\boldsymbol{v})\varphi_j(\boldsymbol{w})\psi_\ell\left(\boldsymbol{v}^+_{ij}\right)\psi_i\left(\boldsymbol{w}^+_{ij}\right) \tag{6.6}$$
$$\varphi_j(\boldsymbol{v})\Psi^*_{ij}(\boldsymbol{\Omega})\psi_i(\boldsymbol{v}) = \varphi_i(\boldsymbol{v})\Phi^*_{ij}(\boldsymbol{\Omega})\psi_j(\boldsymbol{v}). \tag{6.7}$$

The first equation (6.6) gives

$$\ln\frac{\varphi_\ell}{\psi_\ell} = -\frac{1}{T}\left(\frac{1}{2}m\boldsymbol{v}^2 + E_\ell - \mu\right) \tag{6.8}$$

while the second equation (6.7), after using equation (6.8) gives $T = \mathcal{T}$.

In order to study the stability of this equilibrium we introduce the following functional,

$$L_{\mathrm{A}} = H_{\mathrm{A}} + \frac{E_{\mathrm{A}}}{T} \tag{6.9}$$

with H_{A} and E_{A} given in equations (4.2) and (4.8), respectively.

Proposition 4. *L_{A} is a Lyapounov functional for the present problem.*

Proof. First, we find

$$\frac{\mathrm{d}L_{\mathrm{A}}}{\mathrm{d}t} = \mathcal{C}_{\mathrm{A}} \leqslant 0 \tag{6.10}$$

so that L_{A} is a decreasing function. In fact, from the definition (6.9) we obtain

$$\frac{\mathrm{d}L_{\mathrm{A}}}{\mathrm{d}t} = \sum_\ell \int_{\mathbb{R}^3}\left[\frac{\partial\mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell} + \frac{1}{T}\left(\frac{1}{2}m\boldsymbol{v}^2 + E_\ell\right)\right]\left(\frac{\partial f_\ell}{\partial t}\right)^\sharp \mathrm{d}\boldsymbol{v}. \tag{6.11}$$

After using equation (4.4) and taking into account particle conservation we obtain relation (6.10).

Moreover, since

$$\left(\frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell}\right)^* = -\frac{1}{T}\left(\frac{1}{2}m\boldsymbol{v}^2 + E_\ell - \mu\right) \tag{6.12}$$

as it follows from equations (3.11) and (4.4), by accounting for particle and total energy conservation we can write

$$\begin{aligned} L_{\mathrm{A}} - L_{\mathrm{A}}^* &= H_{\mathrm{A}} - H_{\mathrm{A}}^* + \frac{1}{T}(E_{\mathrm{A}} - E_{\mathrm{A}}^*) \\ &= \sum_\ell \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} - \sum_\ell \int_{\mathbb{R}^3} \mathcal{H}_{\mathrm{A}}(f_\ell^*)\,\mathrm{d}\boldsymbol{v} \\ &\quad + \frac{1}{T}\sum_\ell \int_{\mathbb{R}^3} \left(\frac{1}{2}m\boldsymbol{v}^2 + E_\ell - \mu\right)(f_\ell - f_\ell^*)\,\mathrm{d}\boldsymbol{v} \\ &= \sum_\ell \int_{\mathbb{R}^3} \left[\mathcal{H}_{\mathrm{A}}(f_\ell) - \mathcal{H}_{\mathrm{A}}(f_\ell^*) - \left(\frac{\partial \mathcal{H}_{\mathrm{A}}(f_\ell)}{\partial f_\ell}\right)^* (f_\ell - f_\ell^*)\right]\mathrm{d}\boldsymbol{v} \\ &= \sum_\ell \int_{\mathbb{R}^3} \widehat{\mathcal{H}}_{\mathrm{A}}(f_\ell)\,\mathrm{d}\boldsymbol{v} \end{aligned} \tag{6.13}$$

with $\widehat{\mathcal{H}}_{\mathrm{A}}$ given in equation (4.18). Due to the convexity of $\mathcal{H}_{\mathrm{A}}$ we can conclude that $L_{\mathrm{A}} \geqslant L_{\mathrm{A}}^*$ and L_{A} attains a minimum at equilibrium. □

Finally, we can verify that equation (6.10) is equivalent to Clausius inequality

$$\frac{\mathrm{d}S_{\mathrm{A}}}{\mathrm{d}t} \geqslant \frac{1}{T}\frac{\mathrm{d}E_{\mathrm{A}}}{\mathrm{d}t} \tag{6.14}$$

where $S_{\mathrm{A}} = -H_{\mathrm{A}}$. From $L_{\mathrm{A}} - L_{\mathrm{A}}^* \geqslant 0$ it follows that the quantity $\mathcal{S}_{\mathrm{A}} = S_{\mathrm{A}} - E_{\mathrm{A}}/T$ is always increasing and attains a maximum at equilibrium.

7. Numerical simulations

For the purpose of numerical calculations, we consider a homogeneous and isotropic case and keep the generalization for photons only. Atoms are treated as classical particles. This means that substitution (2.19) is performed, while (2.20) is not applied. A couple of different cases are considered for the functions Φ and Ψ.

By integrating equation (2.6) over $\mathrm{d}\boldsymbol{v}$ we obtain

$$\frac{\mathrm{d}N_\ell}{\mathrm{d}t} = \sum_{i>\ell}(S_{\ell i} + 4\pi B_{\ell i}) - \sum_{i<\ell}(S_{i\ell} + 4\pi B_{i\ell}) \tag{7.1}$$

where

$$N_\ell = \int_{\mathbb{R}^3} f_\ell(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \tag{7.2}$$

is the number density of atoms at level ℓ and

$$S_{ij} = \sum_\ell \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} g\sigma_{\ell j}^{\ell i}(g,\zeta)\big[f_\ell(\boldsymbol{v})f_j(\boldsymbol{w}) - f_\ell\big(\boldsymbol{v}_{ij}^+\big)f_i\big(\boldsymbol{w}_{ij}^+\big)\big]\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \tag{7.3}$$

$$B_{ij} = \alpha_{ij}[N_j\Psi(I_{ij}) - N_i\Phi(I_{ij})] \tag{7.4}$$

are the source terms due to the elastic/inelastic and emission/absorption interactions, respectively.

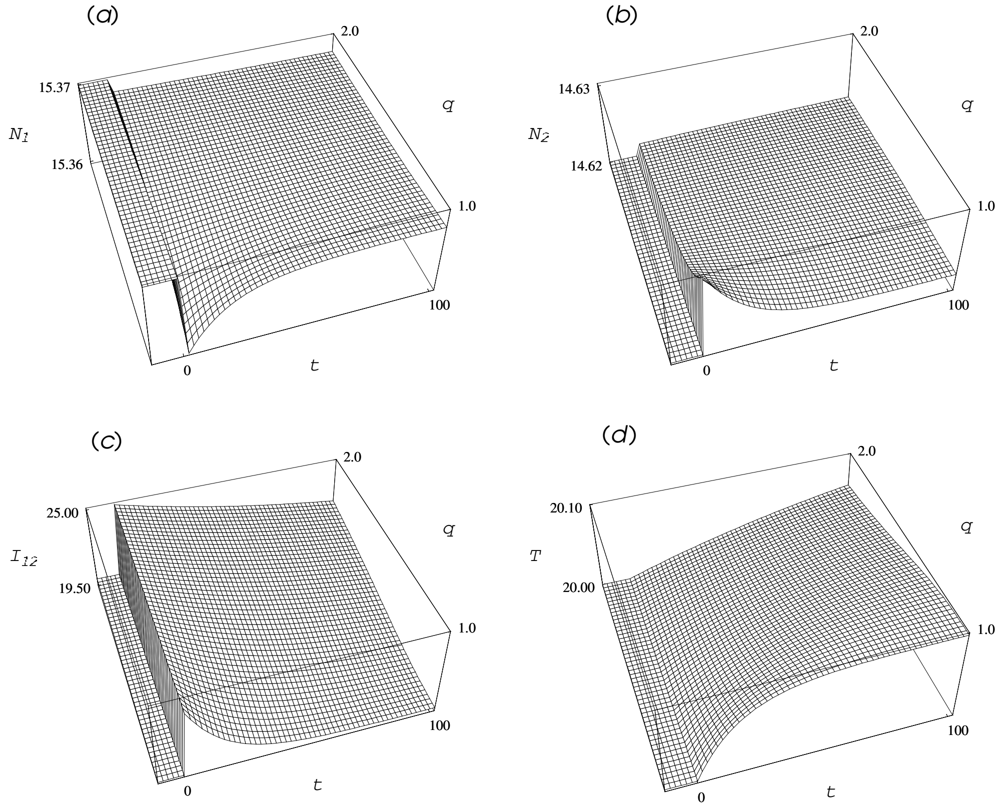

Figure 1. Plot (in arbitrary units) of time evolution of N_1 (*a*), N_2 (*b*), I_{12} (*c*) and T (*d*) versus q, after injection, at $t = 0$, of a flash of photons obeying to the q-Bose statistics.

By multiplying equation (2.6) by $m\boldsymbol{v}^2/2$, by summing over ℓ and by integrating over d$\boldsymbol{v}$ we obtain

$$\frac{3}{2}N\frac{\mathrm{d}T}{\mathrm{d}t} = \sum_j \sum_{i<j} \omega_{ij} S_{ij} \tag{7.5}$$

with the total number density of the mixture $N = \sum_\ell N_\ell$ a constant, and temperature given by

$$T = \frac{1}{3N}\,\mathrm{Tr}\,\mathbb{P} \tag{7.6}$$

where

$$\mathbb{P} = m\sum_\ell \int_{\mathbb{R}^3} (\boldsymbol{v}\otimes\boldsymbol{v})\, f_\ell(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \tag{7.7}$$

is the stress tensor.

Otherwise, by integrating equation (2.17) over dΩ it follows that

$$\frac{\mathrm{d}I_{ij}}{\mathrm{d}t} = \omega_{ij} B_{ij}. \tag{7.8}$$

Suppose now that elastic atom–atom interactions prevail over the inelastic and gas–radiation ones (consistent with the usual local thermodynamic equilibrium approximation). Thus,

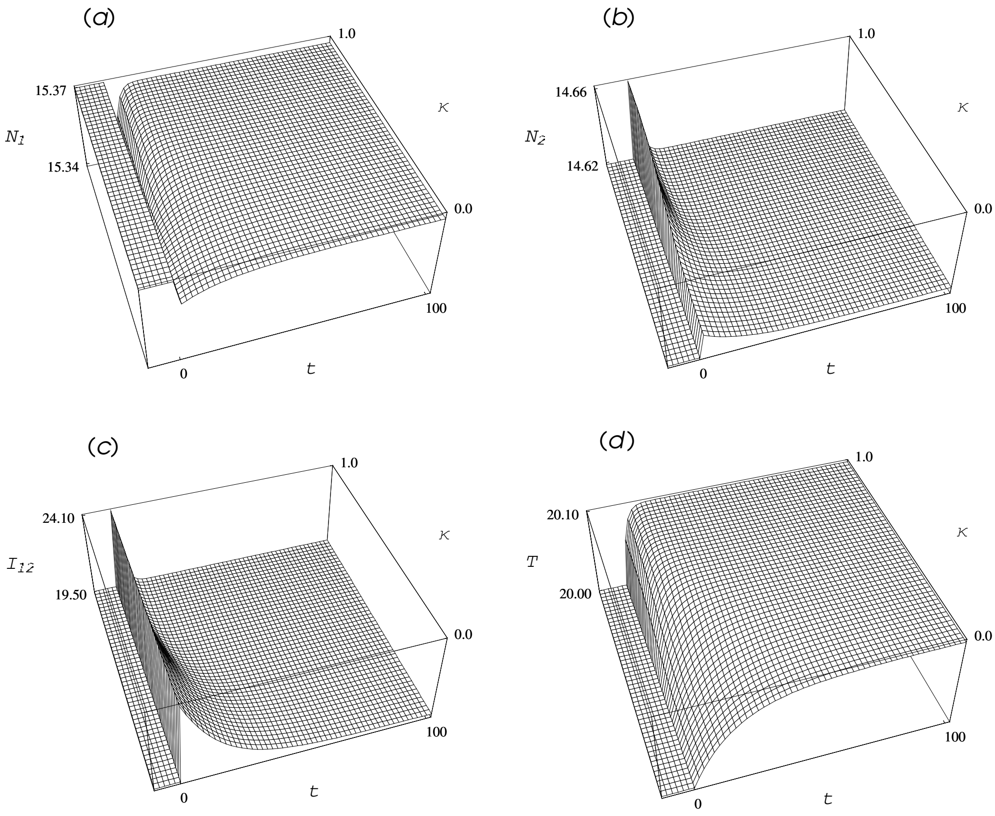

Figure 2. Plot (in arbitrary units) of time evolution of N_1 (*a*), N_2 (*b*), I_{12} (*c*) and T (*d*) versus κ, after injection, at $t = 0$, of a flash of photons obeying to the κ-Bose statistics.

according to the zero-order Chapmann–Enskog approximation we can calculate S_{ij} by means of the Maxwellians

$$f_\ell(\boldsymbol{v}) = N_\ell \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m\boldsymbol{v}^2}{2T}\right). \tag{7.9}$$

It is found that

$$S_{ij} = G_{ij}\left[N_j - N_i \exp\left(-\frac{\omega_{ij}}{T}\right)\right] \tag{7.10}$$

with

$$G_{ij} = \sum_\ell N_\ell \gamma_{\ell j}^{\ell i}(T) \tag{7.11}$$

$$\gamma_{\ell j}^{\ell i}(T) = \left(\frac{m}{2\pi T}\right)^3 \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} g\sigma_{\ell j}^{\ell i}(g,\chi) \exp\left[-\frac{m}{2T}(\boldsymbol{v}^2 + \boldsymbol{w}^2)\right] \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}'. \tag{7.12}$$

Remark that the equilibrium condition is equivalent to $S_{ij} = B_{ij} = 0$, from which follows the relation

$$N_i \exp\left(\frac{E_i}{T}\right) = N_j \exp\left(\frac{E_j}{T}\right). \tag{7.13}$$

Taking into account the total number conservation, we obtain the atom distribution function

$$N_\ell = \frac{N}{Z} \exp\left(-\frac{E_\ell}{T}\right) \tag{7.14}$$

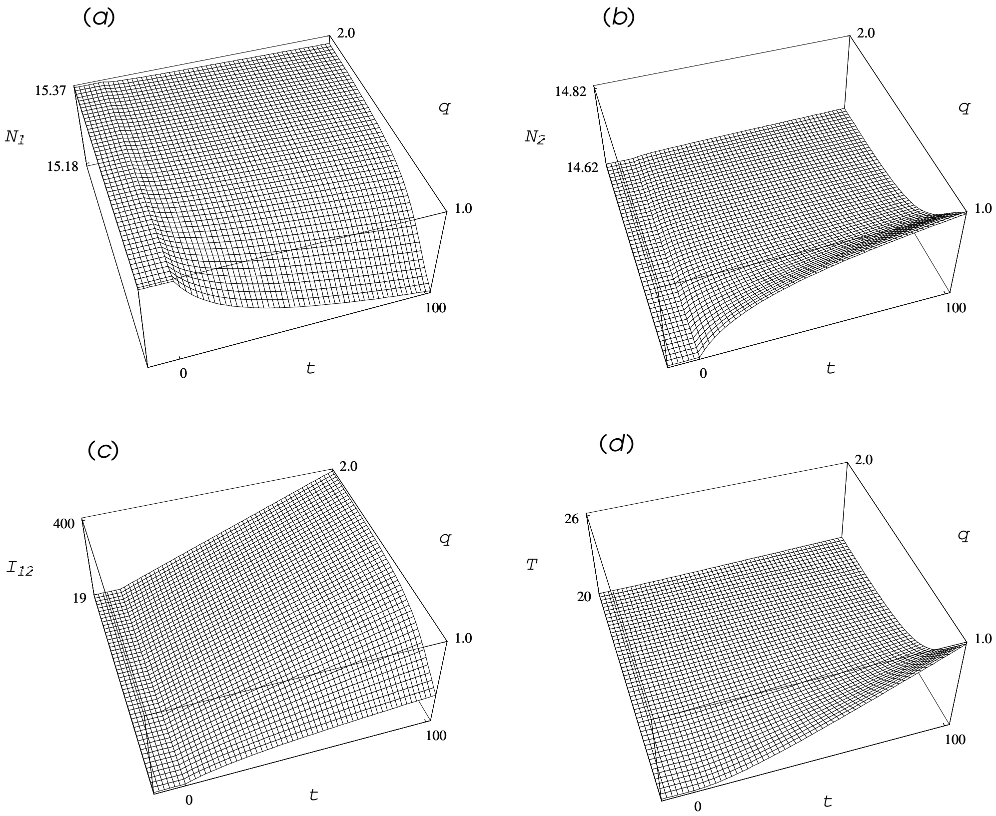

Figure 3. Plot (in arbitrary units) of time evolution of N_1 (*a*), N_2 (*b*), I_{12} (*c*) and T (*d*) versus q, after the introduction, at $t = 0$, of a constant photons source obeying to the q-Bose statistics.

with

$$Z = \sum_\ell \exp\left(-\frac{E_\ell}{T}\right). \tag{7.15}$$

We observe that equations (7.14) and (7.15) assume the standard expression for the equilibrium distributions of the atomic levels. Notwithstanding, because the generalized statistics is kept for the photons, the equilibrium temperature is different with respect to the case in which photons also obey to the standard Bose–Einstein statistics. Consequently, the values of the population levels N_ℓ are numerically different.

Equations (7.1), (7.5) and (7.8) constitute a close system for the unknown quantities N_ℓ, T and I_{ij}. They can be integrated by means of numerical calculation, with suitable initial conditions.

In the following, we consider a system of N atoms with $n = 2$ levels and monochromatic photons with frequency ω_{12}. The system is initially in the equilibrium configuration at temperature T^*. From condition $B_{12} = 0$ follows the distribution I^*_{12}.

Two different simulations are taken into account. In the first simulation, a flash of photons is injected at the time $t = 0$, from an isotropic external source at temperature T'. The relaxation of the system to the new equilibrium configuration is depicted versus time in figures 1 and 2, concerning the results of two different generalizations adopted for the photons and described in

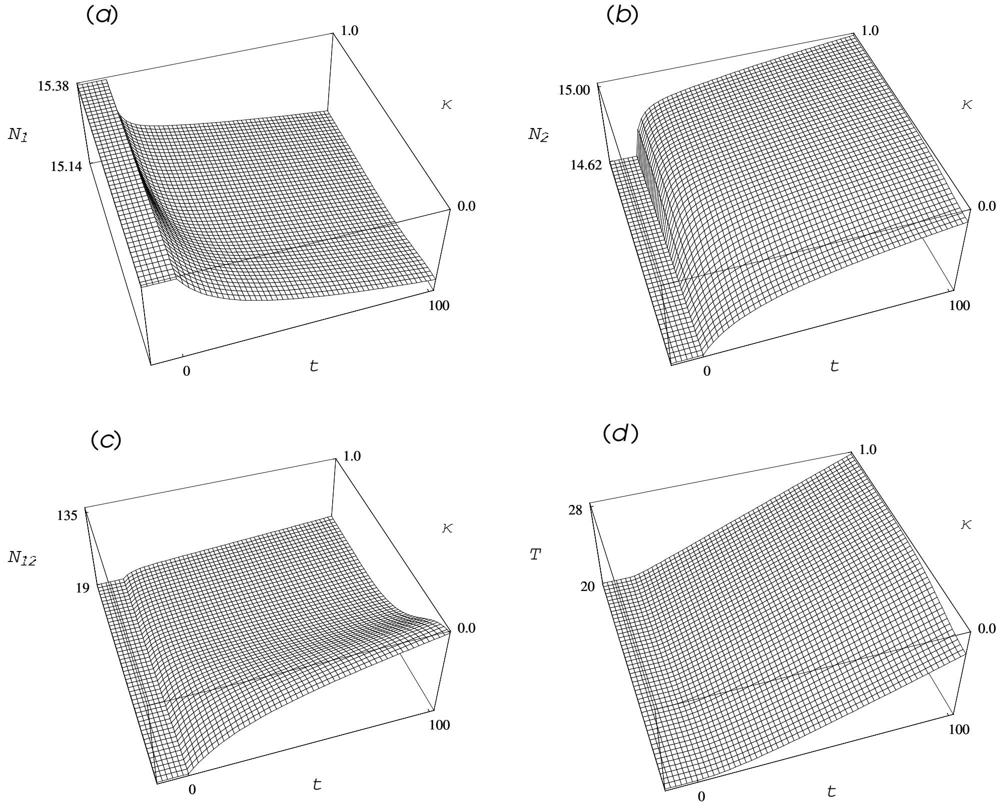

Figure 4. Plot (in arbitrary units) of time evolution of N_1 (*a*), N_2 (*b*), I_{12} (*c*) and T (*d*) versus κ, after the introduction, at $t = 0$, of a constant photons source obeying to the κ-Bose statistics.

appendix B. They take into account an asymptotic inverse power law decay of the distribution function I_{12} with respect to ω_{ij}.

First, we adopt the model proposed by Büyükkiliç *et al* [27, 28]. The time evolution of the physical meaningful quantities N_1, N_2, I_{12} and T versus the deformed parameter q is depicted in figures 1(*a*)–(*d*), respectively.

Then, we adopt the model proposed by Kaniadakis [7]. The time evolution of the physical meaningful quantities N_1, N_2, I_{12} and T versus the deformed parameter κ is depicted in figures 2(*a*)–(*d*), respectively.

In the second simulation, a constant and isotropic external photon source is inserted at the time $t = 0$. Again, both the q-deformation and the κ-deformation are considered. The time evolution of the same physical quantities N_1, N_2, I_{12} and T versus the deformed parameter q (figures 3(*a*)–(*d*)) and versus the deformed parameter κ (figures 4(*a*)–(*d*)) is plotted.

A comparison between the q-deformed case and the κ-deformed case leads to the following considerations. With respect to the classical case, the more the system is κ-deformed, the steeper the evolution is. On the contrary, the more the system is q-deformed the softer the evolution is.

We remark that this opposite behaviour between the q-deformed case and the κ-deformed case is a consequence of the choice $q > 1$. Such a choice arises from the observation that, in the present problem, for values of $q < 1$ a cut-off in the energy spectrum is required; see equation (B.1) in appendix B.

Moreover, from the second simulation we observe that, as well known for the non-deformed case, in presence of a constant pumping of photons, the two number densities reach asymptotically, as $t \to \infty$, to the same value. Such a feature is preserved also in the deformed cases but with a different time-scale.

8. Conclusions

In order to improve the knowledge of the generalized statistical mechanics and generalized kinetic theory, we have investigated, in the present paper, a system of atoms and photons obeying very general statistics. The kinetic equations have been obtained, in the extended Boltzmann picture, through the introduction of characteristic departure and arrival functions for atoms and photons. In the space homogeneous case we have studied the equilibrium configuration. Such equilibrium is given by a modified Planck's law for the radiation and by a generalized distribution function for the atoms. By means of Lyapounov's theory we have studied the stability of this unique equilibrium configuration which maximizes the entropic functional of the system.

In the second part of this paper, we have considered a homogeneous and isotropic case keeping the generalization for photons only. Atoms are treated as classical particles. Under suitable assumptions on the relaxation times of the various interaction processes, according to the zero-order Chapmann–Enskog approximation, we have obtained a close system of macroscopic equations for the unknown dynamical quantities: the number density of atoms N_ℓ at energy E_ℓ, the temperature of the system T and the intensity I_{12} at frequency ω_{12}.

We have shown the results of some numerical simulations for a system with $n = 2$ energy levels. In the first simulation the system, initially at equilibrium at the temperature T^*, is perturbed by a flash of photons injected by a external source at a different temperature T'. The relaxation of the system to the new equilibrium state for the physically relevant quantities N_1, N_2, T and I_{12} has been plotted. In the second simulation the system, initially at equilibrium at the temperature T^*, is disturbed by a constant source of photons. The time evolution of the same physically relevant quantities has been plotted.

Appendix A

For the sake of completeness, we report in this appendix on the classical kinetic equations for atoms and photons, as known in the literature.

The distribution function of atoms A_ℓ obeys to the following system of Boltzmann equations

$$\frac{\partial f_\ell}{\partial t} + \boldsymbol{v} \cdot \nabla f_\ell = J_\ell(\boldsymbol{v}) + \widetilde{J}_\ell(\boldsymbol{v}) \tag{A.1}$$

where on the right-hand side of equation (A.1) $J_\ell(\boldsymbol{v})$ and $\widetilde{J}_\ell(\boldsymbol{v})$ describe the contribution due to the atom–atom interaction and the contribution due to the atom–photon interaction, respectively.

In detail, for elastic/inelastic interactions between atoms we have

$$J_\ell(\boldsymbol{v}) = \sum_j \sum_{i\leqslant j} J^{(1)}_{ij\ell}(\boldsymbol{v}) + \sum_j \sum_{i\leqslant j} J^{(2)}_{ij\ell}(\boldsymbol{v}) + \sum_k \sum_{j\geqslant \ell} J^{(3)}_{jk\ell}(\boldsymbol{v}) + \sum_k \sum_{i\leqslant \ell} J^{(4)}_{ik\ell}(\boldsymbol{v}) \tag{A.2}$$

with

$$J^{(1)}_{ij\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} g\sigma^{\ell i}_{\ell j}(g,\zeta)\big[f_\ell\big(\boldsymbol{v}^+_{ij}\big) f_i\big(\boldsymbol{w}^+_{ij}\big) - f_\ell(\boldsymbol{v}) f_j(\boldsymbol{w})\big]\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \tag{A.3}$$

$$J^{(2)}_{ij\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} g\sigma^{\ell j}_{\ell i}(g,\zeta)[f_\ell(\boldsymbol{v}^-_{ij})f_j(\boldsymbol{w}^-_{ij}) - f_\ell(\boldsymbol{v})f_i(\boldsymbol{w})]\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \qquad \text{(A.4)}$$

$$J^{(3)}_{jk\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} g\sigma^{kj}_{k\ell}(g,\zeta)[f_j(\boldsymbol{v}^-_{\ell j})f_k(\boldsymbol{w}^-_{\ell j}) - f_\ell(\boldsymbol{v})f_k(\boldsymbol{w})]\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \qquad \text{(A.5)}$$

$$J^{(4)}_{ik\ell}(\boldsymbol{v}) = \int_{\mathbb{R}^3\times S^2} g\sigma^{ki}_{k\ell}(g,\zeta)\big[f_i\big(\boldsymbol{v}^+_{i\ell}\big)f_k\big(\boldsymbol{w}^+_{i\ell}\big) - f_\ell(\boldsymbol{v})f_k(\boldsymbol{w})\big]\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{\Omega}' \qquad \text{(A.6)}$$

where $\sigma^{\ell j}_{\ell i}$ and $\sigma^{\ell i}_{\ell j}$ are the cross sections for forward and backward reactions, describing elastic and inelastic interactions.

In equations (A.3)–(A.6) we have

$$\boldsymbol{v}^\pm_{ij} = \frac{1}{2}\big(\boldsymbol{v}+\boldsymbol{w}+g^\pm_{ij}\boldsymbol{\Omega}'\big) \qquad \boldsymbol{w}^\pm_{ij} = \frac{1}{2}\big(\boldsymbol{v}+\boldsymbol{w}-g^\pm_{ij}\boldsymbol{\Omega}'\big) \qquad \text{(A.7)}$$

$$g^\pm_{ij} = \sqrt{g^2 \pm \frac{4}{m}(E_j - E_i)} \qquad g = |\boldsymbol{v}-\boldsymbol{w}| \qquad \text{(A.8)}$$

$$\boldsymbol{\Omega} = \frac{1}{g}(\boldsymbol{v}-\boldsymbol{w}) \qquad \cos\zeta = \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}' \qquad \text{(A.9)}$$

and the two-dimensional unit sphere S^2 is the domain of integration for the unit vector $\boldsymbol{\Omega}'$. The four contributions to $J_\ell(\boldsymbol{v})$ correspond to the cases in which A_ℓ plays the role, in the reaction scheme (1.1), of A_k on the rhs, A_k on the lhs, A_i and A_j, respectively.

For gas–radiation interactions, one can write

$$\widetilde{J}_\ell(\boldsymbol{v}) = \sum_{i>\ell}\int_{S^2}\widehat{J}_{\ell i}(\boldsymbol{v},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} - \sum_{i<\ell}\int_{S^2}\widehat{J}_{i\ell}(\boldsymbol{v},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} \qquad \text{(A.10)}$$

where

$$\widehat{J}_{i\ell}(\boldsymbol{v},\boldsymbol{\Omega}) = f_\ell(\boldsymbol{v})[\alpha_{i\ell} + \beta_{i\ell}I_{i\ell}(\boldsymbol{\Omega})] - \beta_{i\ell}f_i(\boldsymbol{v})I_{i\ell}(\boldsymbol{\Omega}). \qquad \text{(A.11)}$$

By taking into account all the energy levels higher than ℓ, the loss term is due to absorption, while the gain term is due to spontaneous and stimulated emission. The situation is reversed when we consider all the energy levels lower than ℓ.

The kinetic equation for photons p_{ij} with intensity I_{ij} reads [22]

$$\frac{\partial I_{ij}}{\partial t} + \boldsymbol{\Omega}\cdot\nabla I_{ij} = \omega_{ij}\widetilde{J}_{ij}(\boldsymbol{\Omega}) \qquad \text{(A.12)}$$

where

$$\widetilde{J}_{ij}(\boldsymbol{\Omega}) = \int_{\mathbb{R}^3}\widehat{J}_{ij}(\boldsymbol{v},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{v}. \qquad \text{(A.13)}$$

The gain term is due to spontaneous and stimulated emission, while the loss term is due to absorption.

Appendix B

We describe briefly two models used in the numerical simulations. They take into account an inverse power law decay of the distribution function with respect to energy [1–3].

(1) The first model that we consider has been proposed by Büyükkiliç *et al* [27, 28]. Relevant applications to the black-body problem can be found in [12, 13]. Preliminarily, we define

the q-deformed exponential and logarithm

$$\exp_q(x) = [1 + (1 - q)x]_+^{1/(1-q)} \tag{B.1}$$

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \tag{B.2}$$

where $[x]_+ = x$ for $x \geqslant 0$ and $[x]_+ = 0$ for $x < 0$. These q-deformed functions have been introduced in thermo-statistics for the first time in [29].

The characteristic functions Φ and Ψ are given by [15]

$$\Phi(x) = \exp[\ln_q(x)] \tag{B.3}$$

$$\Psi(x) = \exp\left[\ln_q(x) - \ln_q\left(\frac{x}{1+x}\right)\right] \tag{B.4}$$

where $1 \leqslant q < 2$ and for $q = 1$ the standard statistics is recovered. For bosons we observe that the condition $\Psi(+\infty) = +\infty$ does not hold.

The deformed Bose–Einstein distribution function is given by

$$I_{ij} = \frac{\alpha_{ij}}{\beta_{ij}}\left[\frac{1}{\exp_q(-\omega_{ij}/T)} - 1\right]^{-1}. \tag{B.5}$$

(2) The second model that we consider has been proposed by Kaniadakis [7]. Preliminarily, we define the κ-deformed exponential and logarithm

$$\exp_{\{\kappa\}}(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{1/\kappa} \tag{B.6}$$

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \tag{B.7}$$

Remarkably, the κ-exponential satisfies the property $\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(-x) = 1$. The characteristic functions Φ and Ψ are given by [15]

$$\Phi(x) = \exp[\ln_{\{\kappa\}}(x)] \tag{B.8}$$

$$\Psi(x) = \exp\left[\ln_{\{\kappa\}}(x) - \ln_{\{\kappa\}}\left(\frac{x}{1+x}\right)\right] \tag{B.9}$$

where $0 \leqslant |k| < 1$ and for $\kappa = 0$ the standard statistics is recovered. In the case of bosons, the condition for $\Phi(+\infty) = +\infty$ is fulfilled as well. The deformed Bose–Einstein distribution function is given by

$$I_{ij} = \frac{\alpha_{ij}}{\beta_{ij}}\left[\exp_{\{\kappa\}}\left(\frac{\omega_{ij}}{T}\right) - 1\right]^{-1}. \tag{B.10}$$

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